Calculus in the US: 1940-2004

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June 2004

The teaching of calculus in the U.S. has changed greatly during the last 60 years. Some of these changes have followed directly from increases in college and university enrollment. Other changes, especially the "calculus reform" efforts of the late 1980's, were, in part, reactions to the methods that had developed to teach the large number of students after 1965. The approach in this study will be to link the teaching to the enrollments and to examine the texts that were used at different times. Additionally, I will include comments from some other mathematicians and my personal impressions. In particular, the comments on teaching in the 40's and 50's benefited from discussions with Professor Melvin Henriksen, a well-known, now semi-retired, mathematician who publishes in the area of rings of continuous functions. We begin with the enrollments.

1. U.S. college enrollments and the American university system.

- 1946 1,841,000
- 1955 2,379,000
- 1965 3,840,000
- 1975 9,697,000
- 1985 10,863,000
- 1995 14,715,000
- 2002 16,497,000

Source: US Census Bureau

The data for college enrollments refer to total annual enrollment in four different types of post secondary institutions: research universities, liberal arts colleges, comprehensive universities, and community colleges. Since these types differ from those in China, we will present brief definitions of these institutions to help the reader understand U.S. higher education.

Research universities are post-secondary institutions which may be publicly or privately funded and usually have Ph.D. programs in several disciplines. Students at these universities major in one (or more) fields but also take a variety of courses from other disciplines. (That is, they might be required to take courses which are not directly applicable to their major.) Depending on the university, these students frequently attend freshman courses in large lectures. For assistance with course work, they probably meet with teaching assistants. As is common in most colleges and universities in the U.S., they take four or five courses per semester, each course meeting 3-4 hours per week for 14 weeks, for a semester of 42-56 hours.

Liberal arts colleges are four-year institutions that stress a well-rounded education. These colleges are often as competitive as research universities and attract some of the brightest American students. Liberal arts colleges are typically small (with under 2000 students) and private, but offer scholarships for those who cannot afford tuition. One attraction for students is that these colleges offer more individual attention from professors than students usually receive in research universities.

Comprehensive universities may also be privately or publicly funded and typically have graduate programs in some, but not all, fields. Their students come from the local area and the university may be quite strong in some fields and not as strong in others. (For example, Arcadia University's graduate physical therapy program is ranked eighth in the U.S.)

Community colleges have students who attend for two years. These students may or may not go on to a university after their two years. Community colleges cost less money than other choices and have less competitive admissions, so they often attract large numbers of students.

2. 1930-1950

In June 1944, the United States Congress approved the "G.I.Bill" which provided educational benefits for returning veterans. This lead to a substantial increase in the number of students attending college in the late 1940's. Yet, this increase did not immediately lead to a change in the approach to teaching calculus. There were several reasons why calculus teaching did not change. First, the standard teaching load then was 15 hours per week, so faculty did not have time to revise courses. And with such a large teaching load, research mathematicians did not write texts. In addition, "Lots of new people were hired to take care of the big increases in enrollment. We were forced to teach out of texts chosen by the old guard who had prepared solutions of the problems in the texts they had used for years, and did not want to do it again. We had to rise in status to make any changes, and reductions in teaching loads made that easier" in the mid-50's. [9]

The textbooks of this time period were relatively informal by today's standards. That is, the textbooks tended to have discussions rather than formal proofs. The standard text was *Elements of the Differential and Integral Calculus*, by W. A. Granville, as revised by P. F. Smith and W. R. Longley [8]. Granville's approach to the limit is essentially the standard epsilon-delta definition, but with less formal wording: “The variable v is said to approach the constant *l* as a limit when the successive values of v are such that the numerical value of the difference *v-l* ultimately becomes and remains less than any pre-assigned positive number, however small.” The Mean Value Theorem does appear in the text, but at the end of the chapter on differentiation, almost as an afterthought. The applications of differentiation in the text are mostly to geometry, while the applications of integration are to geometry and physics: centroids, fluid pressure, volumes of revolution. Very few exercises are included in this 516 page book, and most of those are very mechanical or procedural. Here is a typical problem:

> It is desired to make an open top box of greatest possible volume from a square piece of tin whose side is a, by cutting equal squares out of the corners and then folding up the tin to form the sides. What should be the length of a side of the squares cut out?

3. 1950-1965

By the late 1940’s to the mid-50’s there were several new texts that added rigor to the teaching of calculus. These included those by Randolph and Kac, Keokeimeister and Johnson, and, of course, Thomas. These were not immediately widely used. “Sometime in the 50’s John Randolph’s publisher wanted him to delete the definition of absolute value because they feared a loss of adoptions. It took Sputnik (in 1957) to make theory respectable.”[9]

In 1962, Sherwood and Taylor was still used when I first studied calculus. I was expected to be able to prove all of the theorems in the text. (This would be very unusual in a first semester calculus course today in the US.) The most influential text for many years was Thomas’ *Calculus with Analytic Geometry.* Although initially published in 1951, it did not become the standard text until the mid 60’s. (Mel Henriksen [9] comments that the first editions were aimed at too small of an audience and it wasn’t until the 60’s that it was broadened by taking out some of the comments that only the brightest students could follow.) Thomas’ Calculus is now in its 10^{th} edition. We take a closer look at the 1953 edition [15], reprinted in 1983 and called the “Classic Edition.”

The definition of the limit is motivated by first introducing the derivative, then giving the epsilon-delta definition. The Mean Value Theorem is introduced early in the exposition of applications and then is applied to curve sketching. The text introduces partial derivatives as a special case of directional derivatives. Most applications in the text are to physics, but the text does include marginal cost. In my opinion, this edition of the text is not easily read. A comparison with the 4^{th} Edition, published in 1968 [14] shows many changes, including improved exposition. This is especially noticeable in the introduction of the limit, which in the 4^{th} edition is clearer, has better diagrams, and has exercises asking students to construct diagrams illustrating the limits as well as finding the appropriate deltas. Here is a typical problem from the classic edition:

An open rectangular box is to be made from a piece of cardboard 8 inches wide and 15 inches long by cutting out a square from each corner and binding up the sides. Find the dimensions of the box of largest volume.

4. 1965-1975

This was the period of the largest increase in enrollments in colleges and also the time when more students entered college having already studied calculus in high school. We begin with a description of high school calculus and then outline the reactions to increased enrollments.

The Advanced Placement (or "AP") Calculus Exam is a national exam for those students who have studied calculus in high school. Most colleges give AP credit or placement for a sufficiently high score on the exam. Although the AP Calculus exam was first given in the mid-50's, it was not until the mid to late 60's that many students took the exam. In fact, there are really two exams: the AB, which covers one (US) semester's material, and the BC, which covers two semesters (through infinite series). The AP exam continued to grow until the present time (2004) when most American mathematics majors come to college knowing at least one semester of basic calculus.

There were many reactions to the increased enrollments and changes in the make-up of the student body:

- class sizes increased (at least at large state universities);
- the size of the calculus texts increased;
- most text books imitated Thomas;
- universities added honors sections of calculus for the best students;
- some universities increased the number of hours per week in a calculus course;
- universities added specialized calculus classes: "Calculus for Business Students" and "Calculus for Biology Students;"
- a few experimental calculus texts appeared (but did not sell well).

The effect of the increased class sizes and texts will be discussed in the next section. Let me comment here on the honors sections of calculus and the experimental texts.

As less and less theory was taught to college students taking calculus, many colleges organized honors sections of calculus. At Swarthmore College, one of the best liberal arts colleges in the U.S., the honors sections of calculus in the early 1970's used *Calculus* by T. Apostol while its regular sections used Thomas' *Calculus with Analytic Geometry*. The text by Apostol [2] is a rigorous introduction to analysis. Other universities which instituted honors sections used the same text as the regular sections, but either covered more material or covered the same material in more depth

There were some interesting experiments in the teaching of calculus during this period. Two of the most interesting were the texts by Henriksen and Lees [10] and Greenspan and Benney [7].

Single Variable Calculus by Henriksen and Lees was designed to be used with a computer lab. With that in mind, it first defined convergence of sequences and then general limits in terms of sequences. The text stressed numerical approximations and contained many theoretical exercises. A typical example: Explain why the following is true or give a counterexample: If f is differentiable on (a,b), then there exists some c in (a,b) with f(b)-f(a)=f'(c)(b-a). Although this problem is very theoretical, the text differs significantly from Aposotol in that it tries to give (what was then) a modern approach to the material. The approach of Greenspan and Benney's *Calculus: An Introduction to Applied Mathematics* was to use numerical examples but also to introduce modeling.

5. 1975-1987

By the mid-1980's the number of students attending college in the U.S. was at an all time high. The number of students in each calculus class (in large universities) had also increased substantially. As a result, the number (and percentage) of students failing calculus increased. In 1987, Lynn Steen, then President of the Mathematical Association of America commented (as quoted in [6]) that "in large universities, fewer than half of the students who begin calculus finish the term with a passing grade."

Calculus texts had changed to reach the new audience of students. *Calculus* by Howard Anton [1] was one of the standard texts. His approach to limits is typical of the texts of the time. There is a ten page intuitive introduction to limits, followed by ten pages of manipulation, followed by ten pages of formal work with the (now optional) epsilon-delta definition. The text has a total of 1300 pages. Yet, it is remarkably well-written. Here is a typical problem.

> A sheet of cardboard 12 inch square is to be used to make an open box by cutting squares of equal size from the corners and folding the sides. What size square should be cut to obtain a box with largest possible volume?

Beginning in 1985 there was an effort to change both the content and pedagogy of calculus in American universities. There were many reasons why many began to believe it was necessary to "reform" calculus. Some of these reasons were:

- To recognize new technology;

- To acknowledge that present methods were leading to failure of many students;
- To listen to what client disciplines needed;
- To rethink textbooks;
- To review and renew teaching methods;
- To consider reducing class sizes.

The process of reform is worth considering. In 1985, Steve Maurer and Ronald Douglas chaired a special session at the annual meeting of the American Mathematical Society. That lead to two conferences: the conference at Tulane University: Toward a Lean and Lively Calculus in 1986 and the Washington, DC Conference: Calculus for a New Century: A Pump not a Filter, in 1987. (The proceedings of these conferences have been published by the Mathematical Association of America in [4] and [5].) Then in 1987, the National Science Foundation announced its Program for Calculus. The NSF Proposal stated in part:

> “There is a need for revision and renewal in the calculus curriculum.” They suggested that the focus of these efforts should be on “raising students’ conceptual understanding, problem solving skills, analytical and transference skills, while implementing new methods to reduce tedious calculations.” (As quoted in [6].)

This exposition of the need to reform is often taken as the definition of “Reform Calculus,” and so in this paper when we speak of reform calculus, we mean the efforts described by the NSF proposal.

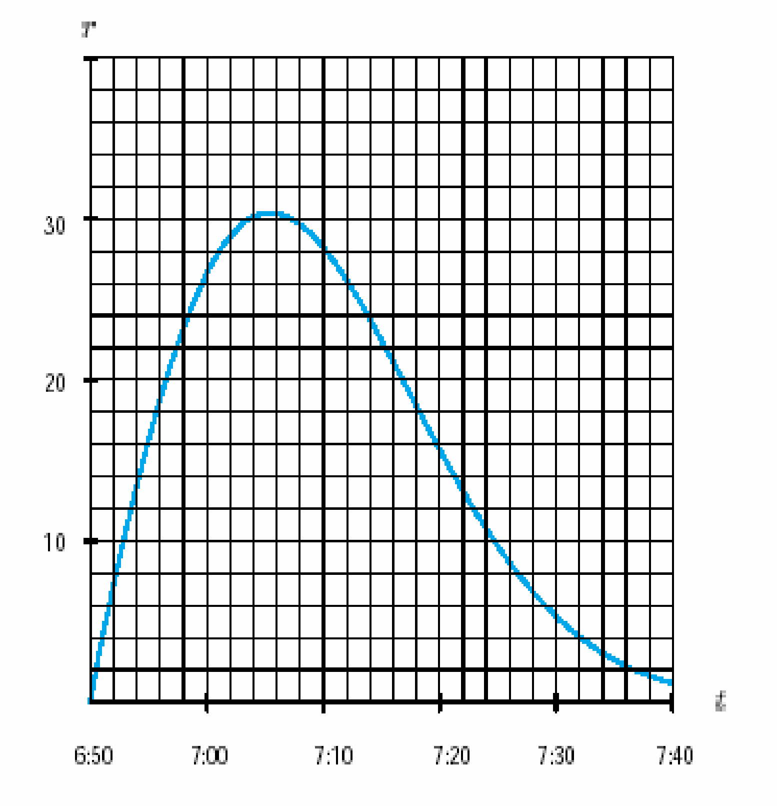
The NSF put much money into the process of reform, and perhaps as a result, many fine mathematicians seriously rethought the entire calculus curriculum. Here is the list of the NSF sponsored programs, ranked by the largest percentage used on other campuses [6])

- Harvard Calculus Consortium (Hughes-Hallett, Gleason,.)
- Duke University (Moore and Smith)
- St Olaf College (Ostebee and Zorn)
- University of Iowa (Stroyan)
- Clemson University (Kenelly)
- University of California –Berkeley (Treisman)
- Purdue University (Dubinsky)
- University of Illinois (Uhl)

These programs produced materials which were used at other colleges. The text developed by the Harvard Calculus Consortium [11] was in many ways truly original. It contained many very different problems. It introduced the “Rule of 4”: every concept is presented graphically, verbally, numerically, algebraically. Yet, the first edition did not include Mean Value Theorem (and this omission became a lightening rod for critics of reform.) The text stresses conceptual understanding. Here are two typical problems from the text. (It is interesting to note that the book does not contain the standard optimization problem that had been used in the US for more than 50 years.)

> Let f(t) be the centimeters of rainfall that has fallen since midnight, where t is in hours. Interpret the following in practical terms, giving units. (a) f(10)=3.1 (b) $f^{-1}(10)=16$ (c) $f'(8)=.04$ (d) $(f^{-1})'(5)=2$

> The graph below represents the rate, r, in arrivals per minute at which students line up for breakfast. The first students arrive at 6:50 and the line opens at 7:00. When the line opens at 7:00, students enter at a constant rate of 20 students per minute. Estimate: (a) the length of the line at 7:00. (b) the rate at which the line is growing at 7:10. (c) the time at which the line has maximum length. (d) the time at which the line disappears.



6. 1987-2004.

The reform efforts initially polarized the mathematical community. Many felt the reforms were long overdue, while others thought that reform was going in the wrong direction. But both sides realized that it was necessary to asses whether or not reform calculus was successful. Many papers in this direction have been published in the last 10 years. This research is summarized in *Changing Calculus. A Report on Evaluation Efforts and National Impact from 1988-1998*, by Susan L. Ganter [6]. Among Ganter's conclusions are:

- Most faculty believe that the "old" way of teaching calculus was ineffective; however there is much debate about whether we are moving in the right direction.

- The relative success or failure of reform efforts is not necessarily dependent upon what is implemented, but rather how, by whom, and in what setting.

- The reform effort has motivated many (often controversial) conversations about the way in which calculus is taught; these conversations are widespread and continuous and have resulted in a renewed sense of importance about undergraduate mathematics education.

- Students (and others) are concerned that their computational skills are no longer as strong as they once were.

- Reports from some institutions indicate that reform students are enrolling in more non-required mathematics courses beyond calculus than their traditional peers, implying that perhaps calculus reform generates more interest in mathematics.

- Overall, 98 of 111 studies (88%) concluded that the impact of reform efforts on at least one measure of student improvement is positive.

- Computers can help students succeed in calculus, with at least 3 institutions observing a remarkable reduction in the number of students failing the course.

My own opinion is that reform calculus has had a tremendous impact on how calculus is and will be taught. For example it has been reported that in 1995, 32% of students were involved in reform experiments. The University of Michigan website indicates that they have 51 sections of Calculus I, all being taught in a reform style. This is more evidence that reform is here to stay. Reform ideas have affected even mainstream calculus text books: Stewart's *Calculus*, which may be the current best selling text, has many problems which use the Rule of 4. And Stewart has also written another text: *Calculus: Concepts and Contexts,* which differs in style, but not approach from the Harvard Consortium's *Calculus*. There are some people who are still trying to deny the effect of technology on teaching calculus. These people will eventually retire.

There is one rather surprising trend during the period from the 1980's to the present that deserves some thought. It is the fact that although overall enrollments in colleges continued to increase, enrollment in mathematics courses did not follow that trend. In particular, enrollments in calculus courses first increased from 590,000 in 1980 to 647,000 in 1990 but then decreased to 538,000 in 1995 before rebounding to 570,000 in 2000 [12]. There are several reasons for this decline. As was mentioned above, more students now take calculus in high school. It is also true that for several disciplines, such as computer science, calculus has been replaced by other courses which may or may not be taught by the mathematics department. Unfortunately, at some universities, "part of the decrease in enrollments has resulted from other department's perceptions that our courses are inappropriate for their needs [3]." This is more evidence of the need to constantly rethink what and how we teach.

7. Final Comments.

The number of students studying calculus in the US greatly increased between 1965 and 1975. The initial response of American mathematicians was to teach the same material in the same way but at a slower pace. This lead to well-written, but irrelevant texts, to large class sizes, and to the failure of large numbers of students, who were forever turned off to mathematics. It was only in the early 1990's with the reform calculus, that mathematicians began to rethink what, why and how they were teaching. This continued rethinking and renewal has been especially important in light of the decrease in calculus enrollments in the U.S.

It is not yet clear if these reform efforts will eventually become the way calculus is taught by most mathematicians in the US. Yet, clearly the discussion has invigorated the teaching of calculus in the US. It is my hope that as enrollments in colleges increase in China, you can avoid our mistakes and begin thinking about the appropriate directions for calculus teaching in your society.

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